Middle East Technical University

Department of Mechanical Engineering

ME 413 Introduction to Finite Element Analysis

Chapter 5

Two-Dimensional Formulation

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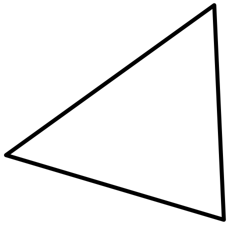
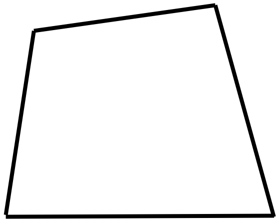
These notes are prepared with the hope to be useful to those who want to learn and teach FEM. You are free to use them. Please send feedbacks to the above email address.

What Are We Going to Learn?

- Compared to 1D, major differences in 2D FEM formulation are
  - application of IBP.
  - master elements and shape functions for triangular and quadrilateral elements.
  - Jacobian transformation.
  - boundary integral evaluation.



Triangular
element



Quadrilateral
element

Model DE in 2D

- Poisson equation in 2D is

$$-\nabla \cdot (a\,\nabla u) = f$$

where $a(x,y)$ and $f(x,y)$ are known functions and

$u(x,y)$ is the unknown.

- For a problem in the xy plane of the Cartesian coordinate system, gradient operator is

$$\nabla \;=\; \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}$$

- In the xy plane Poisson equation becomes

$$-\underbrace{\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}\right)}_{\nabla} \cdot \underbrace{\left(a\frac{\partial u}{\partial x}\vec{i} + a\frac{\partial u}{\partial y}\vec{j}\right)}_{a\,\nabla u} = f$$

$$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(a\frac{\partial u}{\partial y}\right) = f$$

Model DE in 2D

- If function a is constant over the problem domain, Poisson eqn. becomes

$$-a\nabla \cdot (\nabla u) = f \qquad \rightarrow \qquad -\nabla^2 u = g \qquad \rightarrow \qquad -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}\right) = g$$

- Homogeneous form of this equation is called the Laplace's equation

$$-\nabla^2 u = 0 \qquad \rightarrow \qquad -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2}\right) = 0$$

- Poisson equation models many physical phenomena such as
  - potential flow
  - heat conduction
  - groundwater flow
  - transverse deflection of plates
  - electrostatics and magnetostatics

Obtaining Weak Form in 2D

Model DE : $$-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a\frac{\partial u}{\partial y}\right)=f$$

- Weighted residual integral statement of this DE is

$$\int_\Omega w\left[-\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(a\frac{\partial u}{\partial y}\right)-f\right]d\Omega = 0$$

- 2nd order derivatives of u can be reduced to 1st order using the following general equations

$$\int_\Omega w\,\frac{\partial F}{\partial x}\,d\Omega = -\int_\Omega F\,\frac{\partial w}{\partial x}\,d\Omega + \oint_\Gamma wF\,n_x d\Gamma$$

$$\int_\Omega w\,\frac{\partial F}{\partial y}\,d\Omega = -\int_\Omega F\,\frac{\partial w}{\partial y}\,d\Omega + \oint_\Gamma wFn_y d\Gamma$$

Obtaining Weak Form in 2D

$$\int_{\Omega}\left[-w\frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right)-w\frac{\partial}{\partial y}\left(a\frac{\partial u}{\partial y}\right)-wf\right]d\Omega\;=0$$

$$\int_{\Omega} a\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\,d\Omega\;-\;\oint_{\Gamma} wa\frac{\partial u}{\partial x}n_x\,d\Gamma$$

$$\int_{\Omega} a\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\,d\Omega\;-\;\oint_{\Gamma} wa\frac{\partial u}{\partial y}n_y\,d\Gamma$$

- Elemental weak form is

$$\boxed{\int_{\Omega^e} a\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial w}{\partial y}\right)d\Omega\;=\;\int_{\Omega^e} wf\,d\Omega\;+\;\oint_{\Gamma^e} w\left(a\frac{\partial u}{\partial x}n_x+a\frac{\partial u}{\partial y}n_y\right)d\Gamma}$$

$$\underbrace{\left(a\frac{\partial u}{\partial x}n_x+a\frac{\partial u}{\partial y}n_y\right)}_{q_n\text{ : SV of the problem}}$$

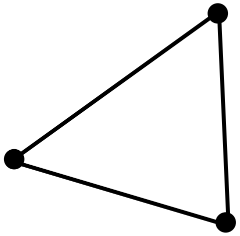
where n_x and n_y are the Cartesian components of the unit outward normal of Γ^e.

2D Formulation (cont'd)

- Approximate solution over an element is

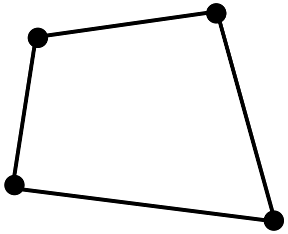
$$u^e = \sum_{j=1}^{NEN} u_j^e \, S_j^e(x,y)$$

- For linear triangular and quadratic elements NEN is 3 and 4, respectively.



3-node triangular element

$NEN = 3$



4-node quadrilateral element

$NEN = 4$

2D Formulation (cont'd)

- To get the i^{th} equation of element e
  - substitute approximate u^e into the elemental weak form and
  - select $w = S_i^e$

$$\int_{\Omega^e} a\left[\frac{\partial}{\partial x}\left(\sum_{j=1}^{NEN} u_j^e S_j^e\right)\frac{\partial S_i^e}{\partial x} + \frac{\partial}{\partial y}\left(\sum_{j=1}^{NEN} u_j^e S_j^e\right)\frac{\partial S_i^e}{\partial y}\right] d\Omega = \int_{\Omega^e} S_i^e f \, d\Omega + \oint_{\Gamma^e} S_i^e \, q_n d\Gamma$$

- Arrange to get

$$\sum_{j=1}^{NEN}\underbrace{\left[\int_{\Omega^e} a\left(\frac{\partial S_j^e}{\partial x}\frac{\partial S_i^e}{\partial x} + \frac{\partial S_j^e}{\partial y}\frac{\partial S_i^e}{\partial y}\right) d\Omega\right]}_{K_{ij}^e} u_j^e = \underbrace{\int_{\Omega^e} S_i^e f \, d\Omega}_{F_i^e} + \underbrace{\oint_{\Gamma^e} S_i^e \, q_n d\Gamma}_{Q_i^e}$$

2D Formulation (cont'd)

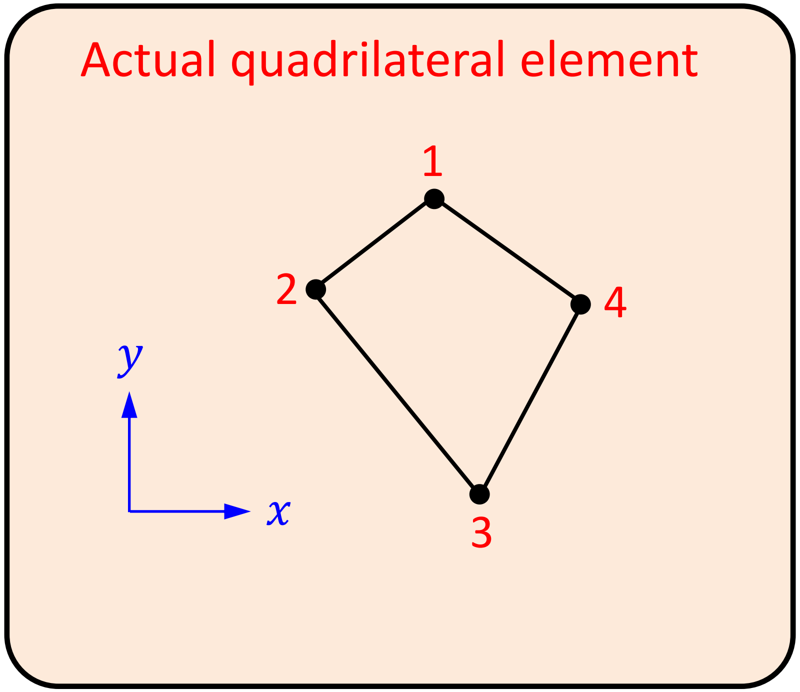
- $NEN \times NEN$ elemental system is

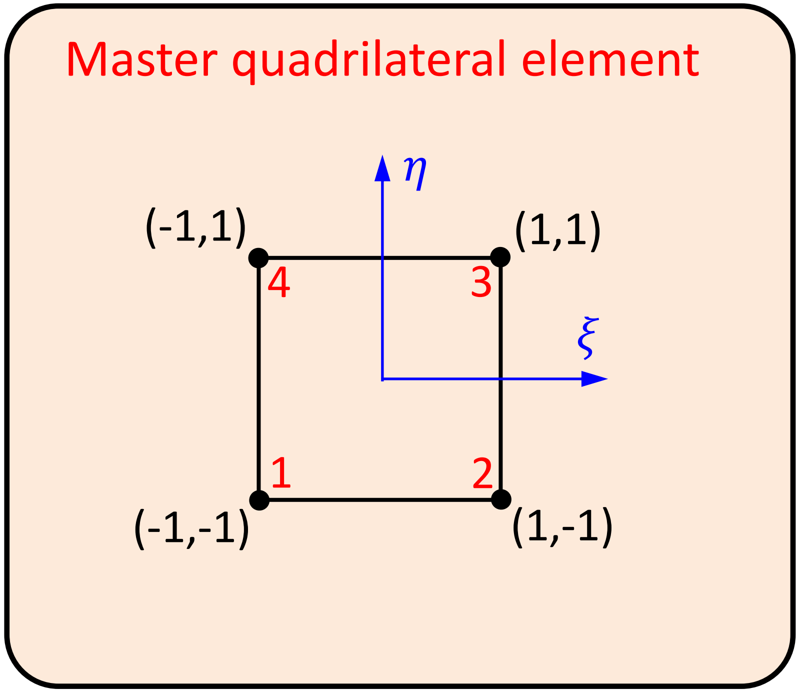
$$[K^e]\{u^e\} = \{F^e\} + \{Q^e\}$$

$$K_{ij}^e = \int_{\Omega^e} a\left(\frac{\partial S_j^e}{\partial x}\frac{\partial S_i^e}{\partial x} + \frac{\partial S_j^e}{\partial y}\frac{\partial S_i^e}{\partial y}\right) d\Omega \qquad F_i^e = \int_{\Omega^e} S_i^e f \, d\Omega \qquad Q_i^e = \oint_{\Gamma^e} S_i^e \, q_n d\Gamma$$

- To evaluate these integrals
  - triangular and quadrilateral master elements will be introduced.
  - shape functions will be written in master element coordinates.
  - 2D Jacobian transformation will be used.
  - GQ integration will be used.

2D Quadrilateral Master Element





- Master quadrilateral element is a square of size 2x2.
- Its nodes are always numbered in a CCW order starting with (-1,-1) corner.
- Nodes of the actual element are also numbered in CCW order. It does NOT matter which node is selected as the first one.

Shape Functions of 2D Quadrilateral Master Element

- General form of Lagrange type 2D shape functions over a 4-node quadrilateral element is

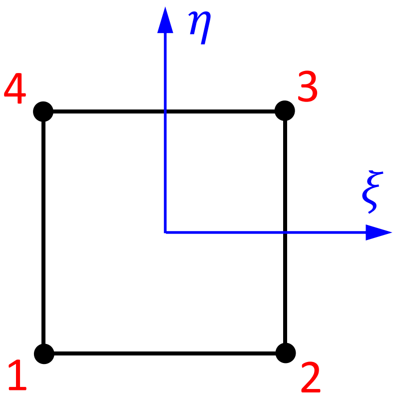
$$S = A + B\xi + C\eta + D\xi\eta$$

- Unknown constants A, B, C and D can be found using the fact that shape functions satisfy the Kronecker-Delta property

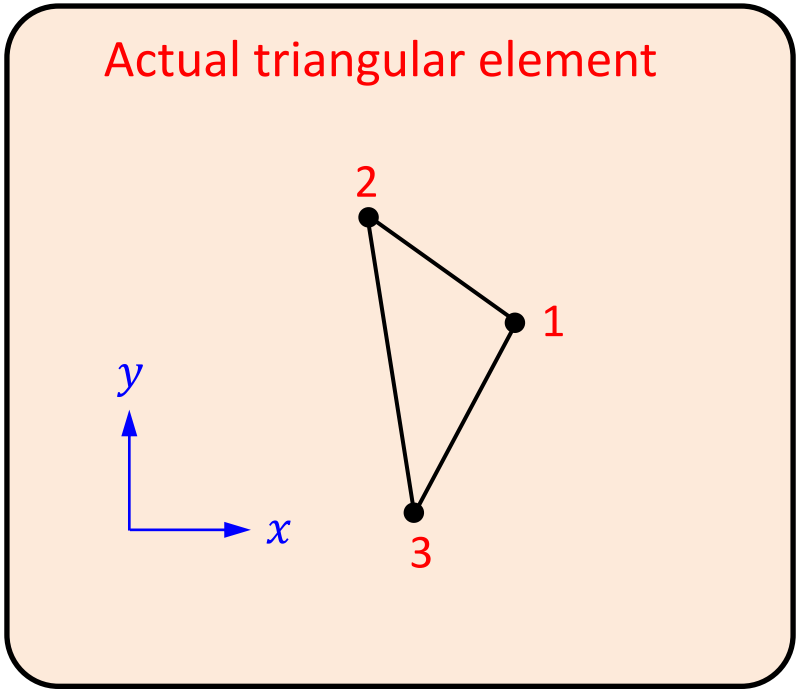
$$S_j(\xi_i,\eta_i) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

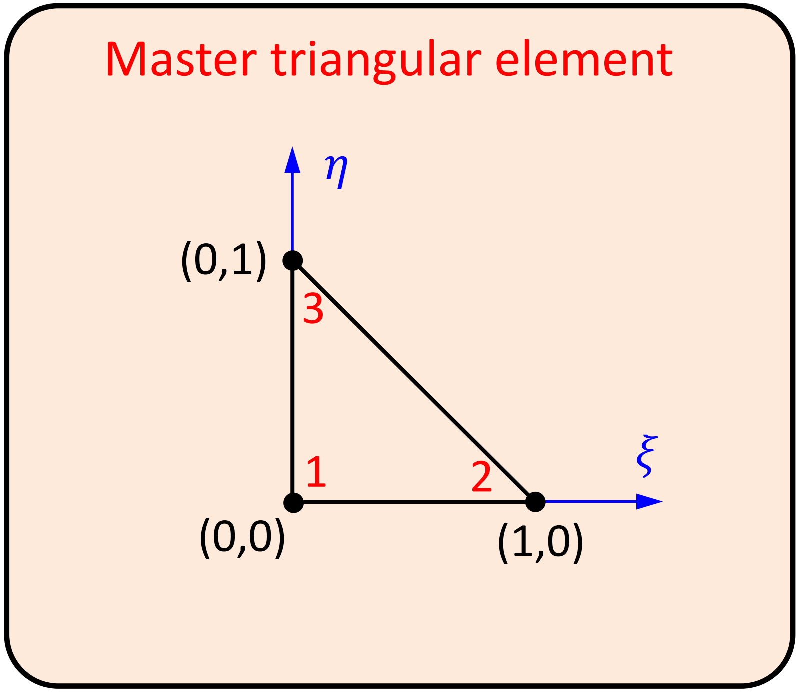
- Shape functions are

$$\boxed{\begin{aligned} S_1 &= \frac{1}{4}(1-\xi)(1-\eta) \\ S_2 &= \frac{1}{4}(1+\xi)(1-\eta) \\ S_3 &= \frac{1}{4}(1+\xi)(1+\eta) \\ S_4 &= \frac{1}{4}(1-\xi)(1+\eta) \end{aligned}}$$



2D Triangular Master Element





- Master triangular element is a right triangle with an area of 0.5.
- Its nodes are always numbered in a CCW order, starting with (0,0) corner.

Shape Functions of 2D Triangular Master Element

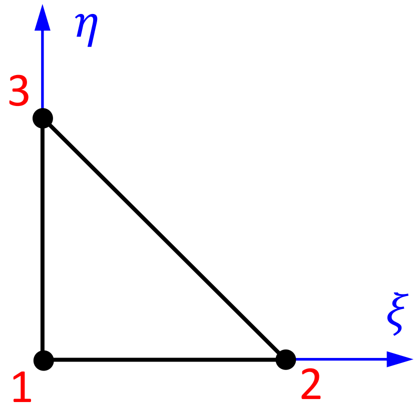
- General form of Lagrange type 2D shape functions over a 3-node triangular element is

$$S = A + B\xi + C\eta$$

- Unknown constants A, B and C can be found using the Kronecker-Delta property of the shape functions

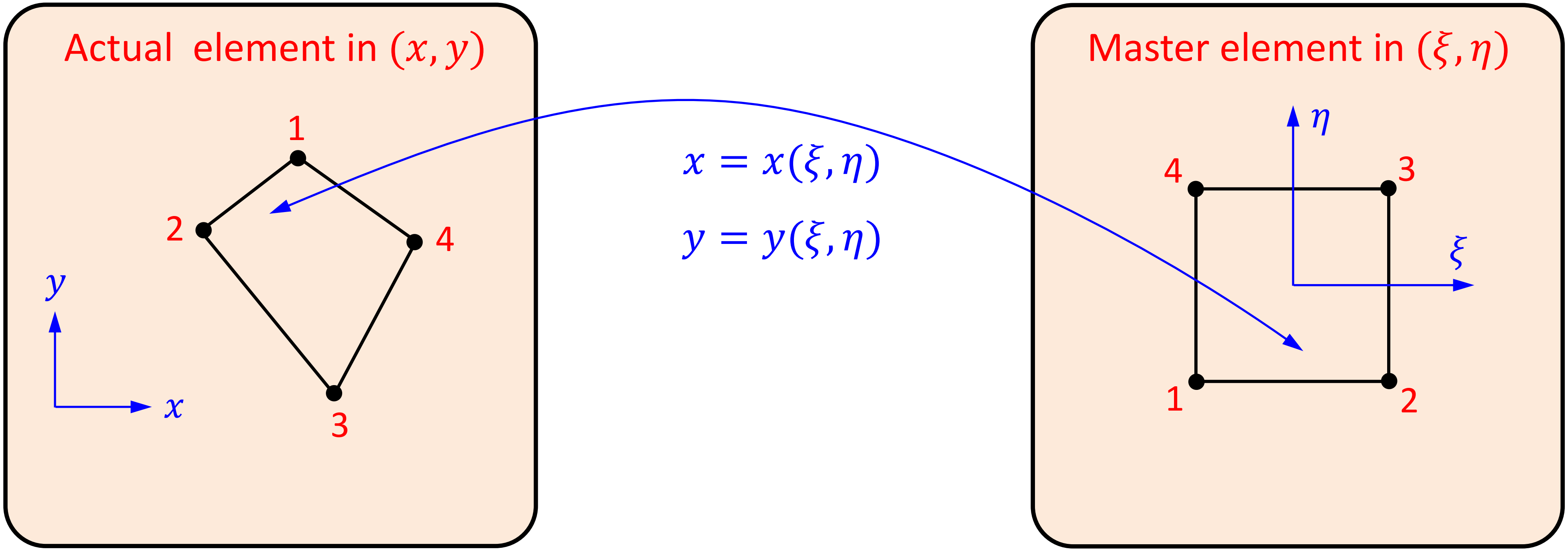
- Shape functions are

$$\boxed{\begin{aligned} S_1 &= 1 - \xi - \eta \\ S_2 &= \xi \\ S_3 &= \eta \end{aligned}}$$



Jacobian Transformation in 2D

- $\partial S/\partial x$ and $\partial S/\partial y$ derivatives appear in the integrals of Slide 5-9.
- These derivatives need to be expressed in terms of $\partial S/\partial \xi$ and $\partial S/\partial \eta$ derivatives.
- This requires the transformation between (x, y) and (ξ, η) coordinates.



Jacobian Transformation in 2D (cont'd)

- Remember that in 1D $x(\xi)$ relation was

$$x = \frac{h^e}{2}\xi + \frac{x_1^e + x_2^e}{2}$$

- This relation can also be expressed as

$$x = \sum_{j=1}^{NEN} x_j^e \, S_j \qquad \rightarrow \qquad x = \frac{1-\xi}{2} x_1^e + \frac{1+\xi}{2} x_2^e$$

- This works due to the Kronecker-Delta property of the shape functions
  - $\xi = -1$ is mapped to $x = x_1^e$
  - $\xi = 1$ is mapped to $x = x_2^e$

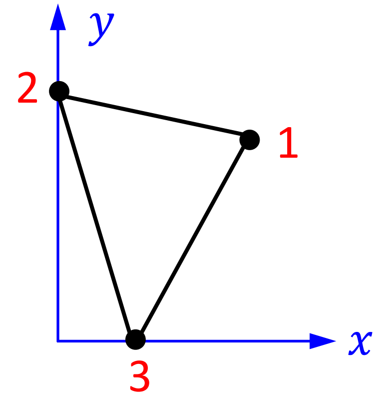
- Same logic can also be used in 2D to get $x(\xi, \eta)$ and $x(\xi, \eta)$ relations.

Jacobian Transformation in 2D (cont'd)

$$x = \sum_{j=1}^{NEN} x_j^e \, S_j \qquad \text{and} \qquad y = \sum_{j=1}^{NEN} y_j^e \, S_j$$

- These can be used for both quadrilateral and triangular elements.
- x_j^e and y_j^e are the coordinates of the corner points of the elements.

e.g. **Example 5.1:** Obtain $x(\xi,\eta)$ and $y(\xi,\eta)$ relations for the following element.



Corner coordinates are

Corner 1 : (5, 6)
Corner 2 : (0, 7)
Corner 3 : (2, 0)

Example 5.1 (cont'd)

$$x = \sum_{j=1}^{3} x_j^e \, S_j \quad = \quad 5(1-\xi-\eta) + 0(\xi) + 2(\eta) = 5 - 5\xi - 3\eta$$

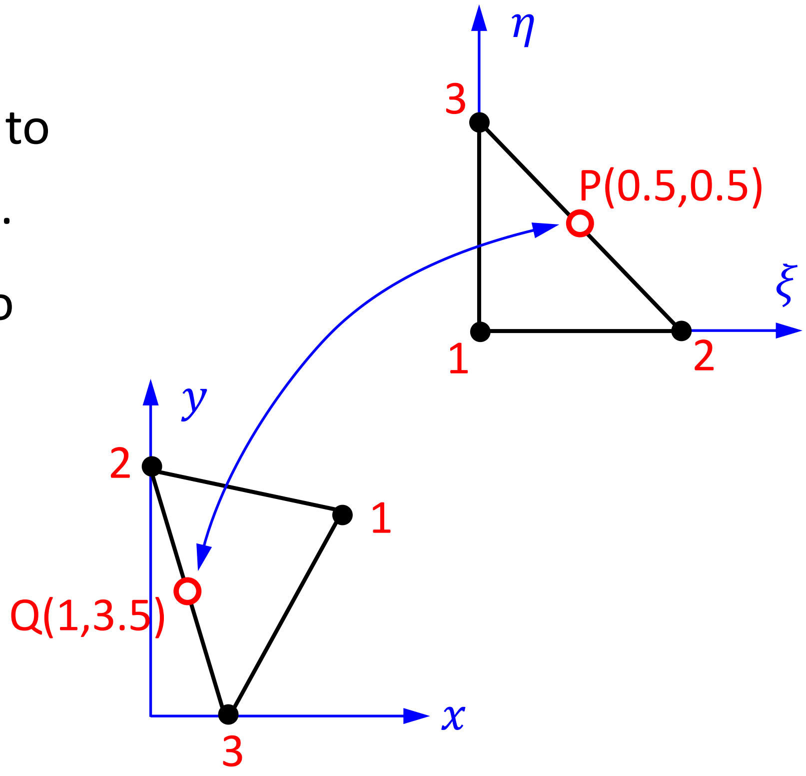
$$y = \sum_{j=1}^{3} y_j^e \, S_j \quad = \quad 6(1-\xi-\eta) + 7(\xi) + 0(\eta) = 6 + \xi - 6\eta$$

- Every point on the master element can be mapped to a point on the actual element using these relations.
- For example point P with $(\xi, \eta) = (0.5, 0.5)$ maps to

$$x = 5 - 5(0.5) - 3(0.5) = 1$$

$$y = 6 + 0.5 - 6(0.5) = 3.5$$

- Both points P and Q are the mid-points of the faces opposite to node 1.



Jacobian Transformation in 2D (cont'd)

- With the link between (x, y) and (ξ, η) coordinates, $\partial S/\partial x$ and $\partial S/\partial y$ derivatives can be linked to $\partial S/\partial \xi$ and $\partial S/\partial \eta$.

$$\frac{\partial S}{\partial \xi} = \frac{\partial S}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial S}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial S}{\partial \eta} = \frac{\partial S}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial S}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\rightarrow \quad \begin{Bmatrix} \dfrac{\partial S}{\partial \xi} \\ \dfrac{\partial S}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}_{\substack{\text{Jacobian matrix} \\ [J^e]}} \begin{Bmatrix} \dfrac{\partial S}{\partial x} \\ \dfrac{\partial S}{\partial y} \end{Bmatrix}$$

- In 1D Jacobian was $J^e = \dfrac{dx}{d\xi}$.
- In 2D Jacobian is a matrix.
- In general $[J^e]$ is different for each element of the FE mesh.

Jacobian Transformation in 2D (cont'd)

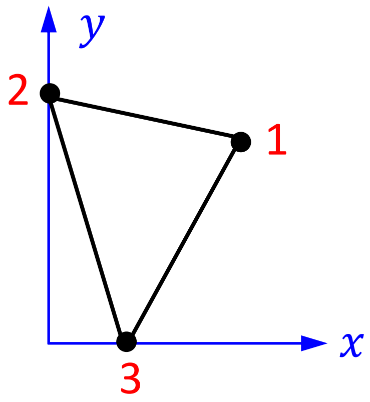
- For the integrals of slide 5-9 what we actually need is

$$\begin{Bmatrix} \dfrac{\partial S}{\partial x} \\ \dfrac{\partial S}{\partial y} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix}}_{\substack{\text{Inverse of the} \\ \text{Jacobian matrix} \\ [J^e]^{-1}}} \begin{Bmatrix} \dfrac{\partial S}{\partial \xi} \\ \dfrac{\partial S}{\partial \eta} \end{Bmatrix}$$

- Since we know x and y as a function of ξ and η, but not the other way, it is NOT practical to calculate $[J^e]^{-1}$ directly.
- Instead we first calculate $[J^e]$ and then take its inverse.

Example 5.2

e.g. **Example 5.2:** Obtain $[J^e]$ and $[J^e]^{-1}$ of the element that we studied in exercise 5.1.



Corner coordinates are

Corner 1 :	(5, 6)
Corner 2 :	(0, 7)
Corner 3 :	(2, 0)

$$[J^e] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum x_j^e \frac{\partial S_j}{\partial \xi} & \sum y_j^e \frac{\partial S_j}{\partial \xi} \\ \sum x_j^e \frac{\partial S_j}{\partial \eta} & \sum y_j^e \frac{\partial S_j}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial S_1}{\partial \xi} & \frac{\partial S_2}{\partial \xi} & \frac{\partial S_3}{\partial \xi} \\ \frac{\partial S_1}{\partial \eta} & \frac{\partial S_2}{\partial \eta} & \frac{\partial S_3}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \end{bmatrix}$$

- This general $[J^e]$ calculation formula applies to both triangular and quadrilateral elements.

Example 5.2 (cont'd)

- For a triangular element shape functions are

$$S_1 = 1 - \xi - \eta \quad , \qquad S_2 = \xi \quad , \qquad S_3 = \eta$$

- For a triangular element derivatives of the shape functions are

$$\begin{bmatrix} \frac{\partial S_1}{\partial \xi} & \frac{\partial S_2}{\partial \xi} & \frac{\partial S_3}{\partial \xi} \\ \frac{\partial S_1}{\partial \eta} & \frac{\partial S_2}{\partial \eta} & \frac{\partial S_3}{\partial \eta} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

- Jacobian of the element is

$$[J^e] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 0 & 7 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -3 & -6 \end{bmatrix}$$

Example 5.2 (cont'd)

- Inverse of the Jacobian matrix is

$$[J^e]^{-1} = \frac{1}{|J^e|}\begin{bmatrix} J^e_{22} & -J^e_{12} \\ -J^e_{21} & J^e_{11} \end{bmatrix}$$

$$|J^e| = J^e_{11}J^e_{22} - J^e_{12}J^e_{21}$$
$$= (-5)(-6) - (1)(-3) = 33$$

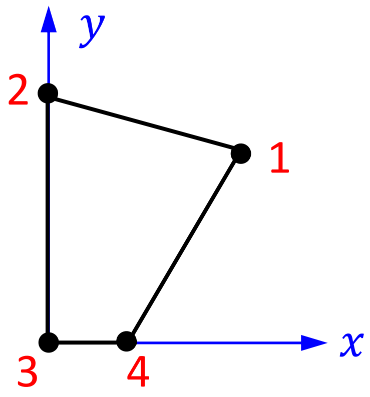
$$[J^e]^{-1} = \frac{1}{33}\begin{bmatrix} -6 & -1 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} -0.182 & 0.030 \\ 0.091 & -0.152 \end{bmatrix}$$

- Note that in 1D J^e was equal to the ratio of actual element's length to master element's length.

- Similarly for a 3-node triangular element $|J^e|$ is equal to the ratio of actual element's area to master element's area. In this exercise the area ratio is $16.5/(0.5) = 33$.

Example 5.3

e.g. **Example 5.3:** Obtain $[J^e]$ and $|J^e|$ of the element shown below.



Corner coordinates are

Corner 1	(5, 6)
Corner 2	(0, 7)
Corner 3	(0, 0)
Corner 4	(2, 0)

$$[J^e] = \begin{bmatrix} \dfrac{\partial S_1}{\partial \xi} & \dfrac{\partial S_2}{\partial \xi} & \dfrac{\partial S_3}{\partial \xi} & \dfrac{\partial S_4}{\partial \xi} \\ \dfrac{\partial S_1}{\partial \eta} & \dfrac{\partial S_2}{\partial \eta} & \dfrac{\partial S_3}{\partial \eta} & \dfrac{\partial S_4}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1^e & y_1^e \\ x_2^e & y_2^e \\ x_3^e & y_3^e \\ x_4^e & y_4^e \end{bmatrix} = \begin{bmatrix} \dfrac{\eta - 1}{4} & \dfrac{1 - \eta}{4} & \dfrac{\eta + 1}{4} & \dfrac{-1 - \eta}{4} \\ \dfrac{\xi - 1}{4} & \dfrac{-\xi - 1}{4} & \dfrac{\xi + 1}{4} & \dfrac{1 - \xi}{4} \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 0 & 7 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}$$

Example 5.3 (cont'd)

$$[J^e] = \frac{1}{4}\begin{bmatrix} 3\eta - 7 & -\eta + 1 \\ 3\xi - 3 & -\xi - 13 \end{bmatrix}$$

- Determinant of $[J^e]$ is

$$|J^e| = J^e_{11}J^e_{22} - J^e_{12}J^e_{21} = \frac{1}{8}(2\xi - 21\eta + 47)$$

- Note that this time both the Jacobian matrix and its determinant are functions of ξ and η.
- Integral of $|J^e|$ over the master element will give the area of the actual element.

$$\int_{\eta=-1}^{1}\int_{\xi=-1}^{1} \underbrace{\frac{1}{8}(2\xi - 21\eta + 47)}_{|J^e|}\, d\xi\, d\eta = 23.5$$

Actual element's area

- This can be generalized as follows which will be used in GQ integration

$$\boxed{\int_{\Omega} dxdy = \int_{\Omega^e} |J^e| d\xi d\eta}$$

(True for both triangular and quadrilateral elements)

Calculation of Integrals Over a Master Element

- Now the integrals of Slide 5-9 can be evaluated on a master element using GQ.

$$K_{ij}^e = \int_{\Omega^e} a \left(\frac{\partial S_j^e}{\partial x}\frac{\partial S_i^e}{\partial x} + \frac{\partial S_j^e}{\partial y}\frac{\partial S_i^e}{\partial y} \right) d\Omega \quad , \qquad F_i^e = \int_{\Omega^e} S_i^e f \, d\Omega$$

Use $x = \sum x_j^e S_j$ and $y = \sum y_j^e S_j$ to convert x and y of function a to ξ and η.

Same for f of F_i^e.

$$\int_{\Omega^e} dxdy = \int_{\Omega^e_{master}} |J^e| d\xi d\eta$$

$$\frac{\partial S^e}{\partial x} = \frac{\partial S}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial S}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial S}{\partial \xi}(J^e)^{-1}_{11} + \frac{\partial S}{\partial \eta}(J^e)^{-1}_{12}$$

$$\frac{\partial S^e}{\partial y} = \frac{\partial S}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial S}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$= \frac{\partial S}{\partial \xi}(J^e)^{-1}_{21} + \frac{\partial S}{\partial \eta}(J^e)^{-1}_{22}$$

Gauss Quadrature Over Quadrilateral Elements

- For a quadrilateral master element both ξ and η change between -1 and 1.
- [-1, 1] are the limits used in 1D GQ integration.
- Therefore for 2D quadrilateral elements 1D GQ tables can be used.

- Consider the evaluation of the following integral using NGP points in both ξ and η directions.

$$I = \int_{\eta=-1}^{1} \int_{\xi=-1}^{1} g \, d\xi \, d\eta$$

In 2D there are NGP^2 GQ points

$$I = \underbrace{\sum_{n=1}^{NGP}}_{\text{Sum for } \eta} \underbrace{\sum_{m=1}^{NGP}}_{\text{Sum for } \xi} g(\xi_m, \eta_n) W_m W_n$$

or

$$I = \sum_{k=1}^{NGP^2} g(\xi_k, \eta_k) W_k$$

Combined sum for ξ and η

$W_k = W_m W_n$

Gauss Quadrature Over Quadrilateral Elements (cont'd)

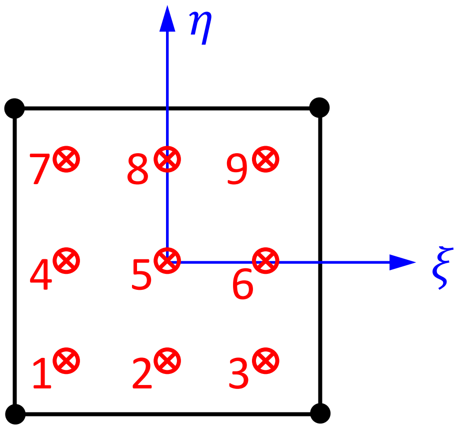
1D GQ Integration

NGP	ξ_k	W_k
1	0.0	2
2	$-1/\sqrt{3}$	1
	$1/\sqrt{3}$	1
3	$-\sqrt{0.6}$	5/9
	0.0	8/9
	$\sqrt{0.6}$	5/9

2D GQ Integration Over Quads

NGP	ξ_k	η_k	W_k
1	0.0	0.0	4
4	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1
	$1/\sqrt{3}$	$-1/\sqrt{3}$	1
	$-1/\sqrt{3}$	$1/\sqrt{3}$	1
	$1/\sqrt{3}$	$1/\sqrt{3}$	1
9	$-\sqrt{0.6}$	$-\sqrt{0.6}$	25/81
	0.0	$-\sqrt{0.6}$	40/81
	$\sqrt{0.6}$	$-\sqrt{0.6}$	25/81
	$-\sqrt{0.6}$	0.0	40/81
	0.0	0.0	64/81
	$\sqrt{0.6}$	0.0	40/81
	$-\sqrt{0.6}$	$\sqrt{0.6}$	25/81
	0.0	$\sqrt{0.6}$	40/81
	$\sqrt{0.6}$	$\sqrt{0.6}$	25/81

9 point GQ over a quad. element



Gauss Quadrature Over Triangular Elements

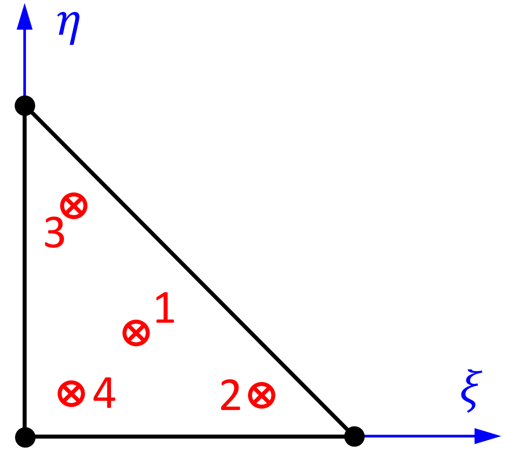
- For a triangular master element limits for ξ and η are [0, 1] and [0, 1-ξ], respectively.

$$\int_{\Omega^e} f \, d\xi d\eta = \int_{\xi=0}^{1} \int_{\eta=0}^{1-\xi} f \, d\eta d\xi$$

- Therefore a new GQ table is necessary.

2D GQ Integration Over Triangles			
NGP	ξ_k	η_k	W_k
1	1/3	1/3	0.5
3	0.5 0.0 0.5	0.0 0.5 0.5	1/6 1/6 1/6
4	1/3 0.6 0.2 0.2	1/3 0.2 0.6 0.2	- 27/96 25/96 25/96 25/96

4 point GQ over a triangular element

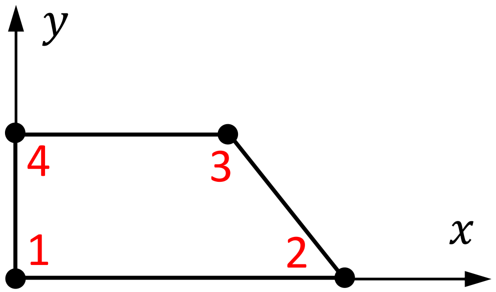


Example 5.4

e.g. **Example 5.4:** Calculate the first entry of the following elemental force vector

$$F_i^e = \int_{\Omega^e} x^2 S_i \, dxdy$$

over the following element using 4 point GQ integration.



Corner coordinates are

Corner 1 : (0, 0)
Corner 2 : (5, 0)
Corner 3 : (3, 2)
Corner 4 : (0, 2)

Example 5.4 (cont'd)

- We need to calculate

$$F_1^e = \int_{\Omega^e} x^2 S_1 \, dxdy$$

- Switching to master element coordinates the integral becomes

$$F_1^e = \int_{-1}^{1}\int_{-1}^{1} x(\xi,\eta)^2 \quad \underbrace{\frac{1}{4}(1-\xi)(1-\eta)}_{S_1} \; \underbrace{|J^e| d\xi d\eta}_{dxdy}$$

- We first need x as a function of ξ and η.

$$x = \sum_{j=1}^{4} x_j^e \, S_j = (0)S_1 + (5)S_2 + (3)S_3 + (0)S_4$$

$$x = (5)\frac{1}{4}(1+\xi)(1-\eta) + (3)\frac{1}{4}(1+\xi)(1+\eta)$$

$$x = \frac{1}{2}(4 + 4\xi - \eta - \xi\eta)$$

Example 5.4 (cont'd)

- Next we need to calculate the Jacobian and its determinant (similar to Slide 5-23)

$$[J^e] = \begin{bmatrix} \frac{\eta-1}{4} & \frac{1-\eta}{4} & \frac{\eta+1}{4} & \frac{-1-\eta}{4} \\ \frac{\xi-1}{4} & \frac{-\xi-1}{4} & \frac{\xi+1}{4} & \frac{1-\xi}{4} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 5 & 0 \\ 3 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4-\eta}{2} & 0 \\ \frac{-1-\xi}{2} & 1 \end{bmatrix}$$

$$|J^e| = \left(\frac{4-\eta}{2}\right)(1) - \left(\frac{-1-\xi}{2}\right)(0) = \frac{4-\eta}{2}$$

- The integral becomes

$$F_1^e = \int_{-1}^{1}\int_{-1}^{1} \underbrace{\left[\frac{1}{2}(4+4\xi-\eta-\xi\eta)\right]^2 \frac{1}{4}(1-\xi)(1-\eta)\frac{4-\eta}{2}}_{g(\xi,\eta)} \, d\xi d\eta$$

$$F_1^e = \int_{-1}^{1}\int_{-1}^{1} g \; d\xi d\eta$$

Example 5.4 (cont'd)

- 4 point GQ over the quadrilateral master element will be

$$F_1^e = g(\xi_1,\eta_1)W_1 + g(\xi_2,\eta_2)W_2 + g(\xi_3,\eta_3)W_3 + g(\xi_4,\eta_4)W_4$$

where points and weights are provided in Slide 5-27

- The result will be

$$F_1^e = 1.3320 + 4.971 + 0.1492 + 0.5569 = 6.9993$$

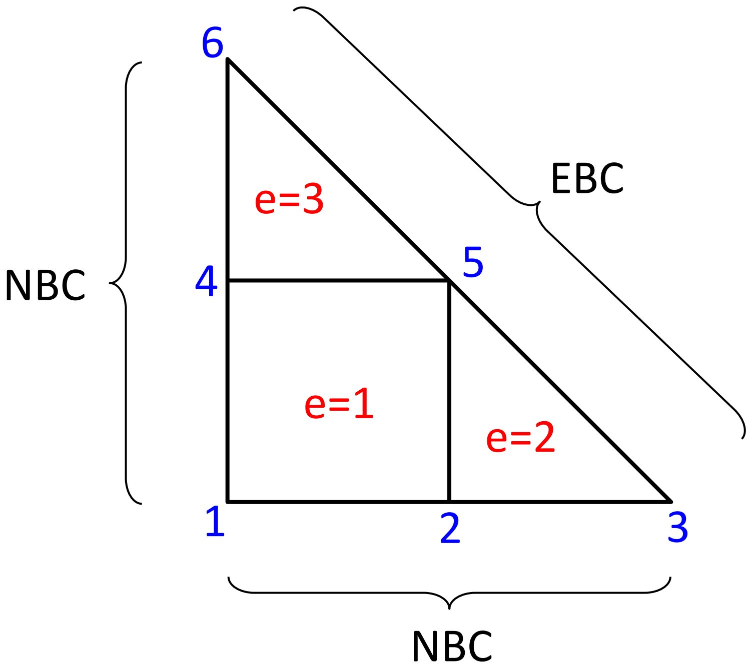
Notes:

- In general $|J^e|$ is a function of ξ and η and it needs to be evaluated at GQ points.
- In this example we did not need $y(\xi,\eta)$ bacause f was not a function of y.
- In this example we did not calculate the inverse of $[J^e]$ because force vector does not contain shape function derivatives. Stiffness matrix calculation will need it.

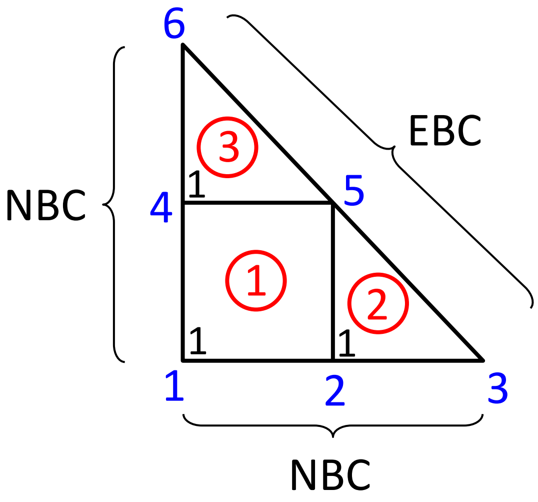
- Is the above result exact? What is the exact value? What will 1 point and 9 point integrations give?

Calculation of $\{Q\}$

- $\{Q\}$ integrals need to be evaluated only for the real boundary faces where NBC or MBC is specified.
- Consider the following problem with a mesh of 3 elements and 6 nodes.



Calculation of $\{Q\}$ (cont'd)



- There are 4 elements faces at NBC or MBC boundaries.
- The assembled global $\{Q\}$ will be (first local node of each element is shown, the others are located in a CCW order)

$$Q = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} \boxed{Q_1^1} \\ \boxed{Q_2^1 + Q_1^2} \\ Q_2^2 \\ \boxed{Q_4^1 + Q_1^3} \\ Q_3^1 + Q_3^2 + Q_2^3 \\ Q_2^3 \end{Bmatrix}$$

- Q_3, Q_5 and Q_6 are not necessary because PVs are known at these nodes.
- Only the circled ones are necessary.

- Note : In this example we do not have an internal node (a node that is not located at a boundary), but for those nodes sum of Q_i^e 's will be zero.

Calculation of $\{Q\}$ (cont'd)

- For the Poisson equation $\{Q^e\}$ is calculated as (Slide 5-9)

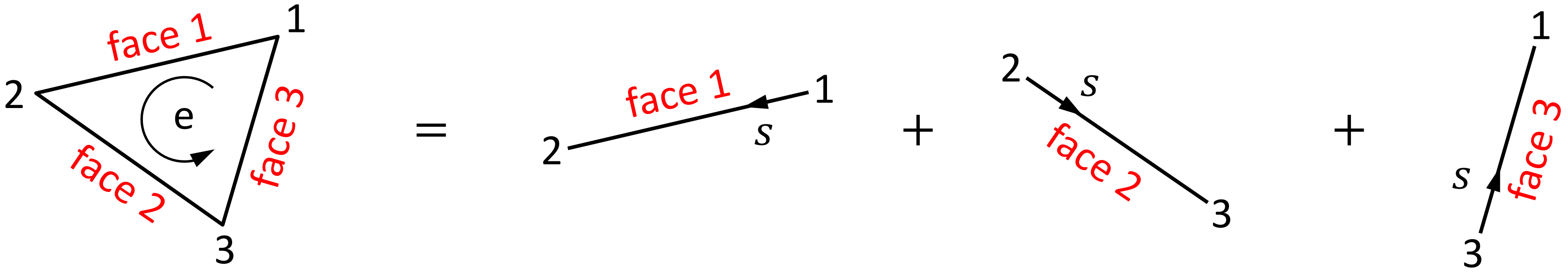
$$Q_i^e = \oint_{\Gamma^e} S_i \, q_n \, ds$$

$$q_n = a\frac{\partial u}{\partial x}n_x + a\frac{\partial u}{\partial y}n_y$$

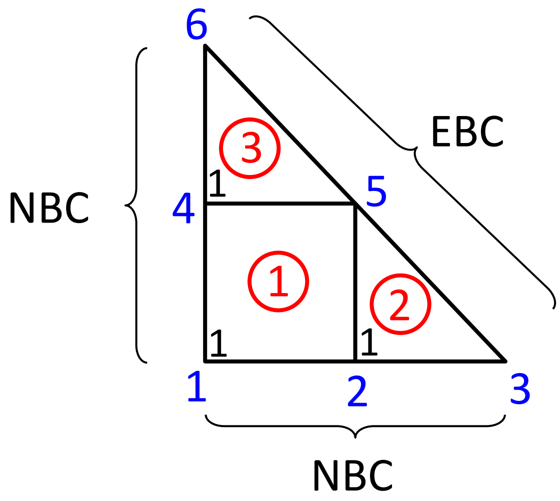
- Γ^e is the boundary of the element and it is composed of NEN straight lines.
- For a triangular element the integral can be decomposed into 3 parts.

$$Q_i^e = \int_{f1} S_i q_n \, ds + \int_{f2} S_i q_n \, ds + \int_{f3} S_i q_n \, ds$$

$f3$: Face 3



Calculation of $\{Q\}$ (cont'd)



- Consider the 3 element problem of Slide 5-34.
- We need $Q_1^1, Q_2^1, Q_4^1, Q_1^2, Q_1^3$

- e=1 :

$$Q_1^1 = \int_{f1} S_1 q_n \, ds + \cancel{\int_{f2} S_1 q_n \, ds} + \cancel{\int_{f3} S_1 q_n \, ds} + \int_{f4} S_1 q_n \, ds$$

S_1 is zero on faces 2 and 3

$$Q_2^1 = \int_{f1} S_2 q_n \, ds + \cancel{\int_{f2} S_2 q_n \, ds} + \cancel{\int_{f3} S_2 q_n \, ds} + \cancel{\int_{f4} S_2 q_n \, ds}$$

No need (f2 is internal)

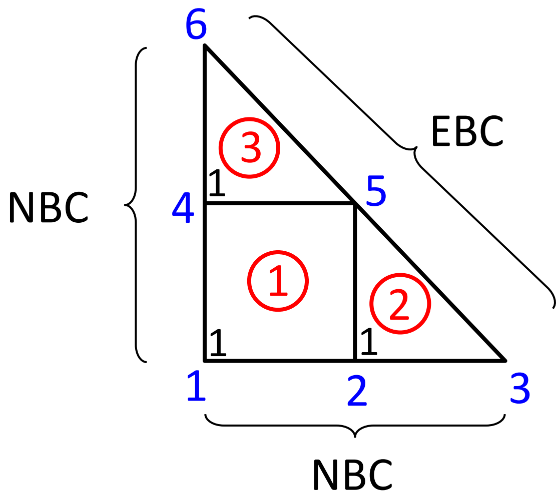
S_2 is zero on faces 3 and 4

$$Q_4^1 = \cancel{\int_{f1} S_4 q_n \, ds} + \cancel{\int_{f2} S_4 q_n \, ds} + \cancel{\int_{f3} S_4 q_n \, ds} + \int_{f4} S_4 q_n \, ds$$

S_4 is zero on faces 1 and 2

No need (f3 is internal)

Calculation of $\{Q\}$ (cont'd)



- e=2 :

$$Q_1^2 = \int_{f1} S_1 q_n \, ds + \cancel{\int_{f2} S_1 q_n \, ds} + \cancel{\int_{f3} S_1 q_n \, ds}$$

S_1 is zero on face 2

No need (f3 is internal)

- e=3 :

$$Q_1^3 = \cancel{\int_{f1} S_1 q_n \, ds} + \cancel{\int_{f2} S_1 q_n \, ds} + \int_{f3} S_1 q_n \, ds$$

No need (f1 is internal)

S_1 is zero on face 2

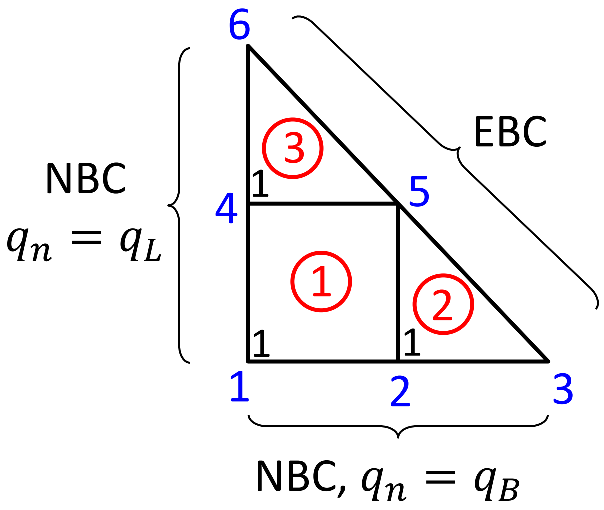
- Conclusion : Boundary integrals need to be calculated only for real boundary faces where NBC or MBC is provided.

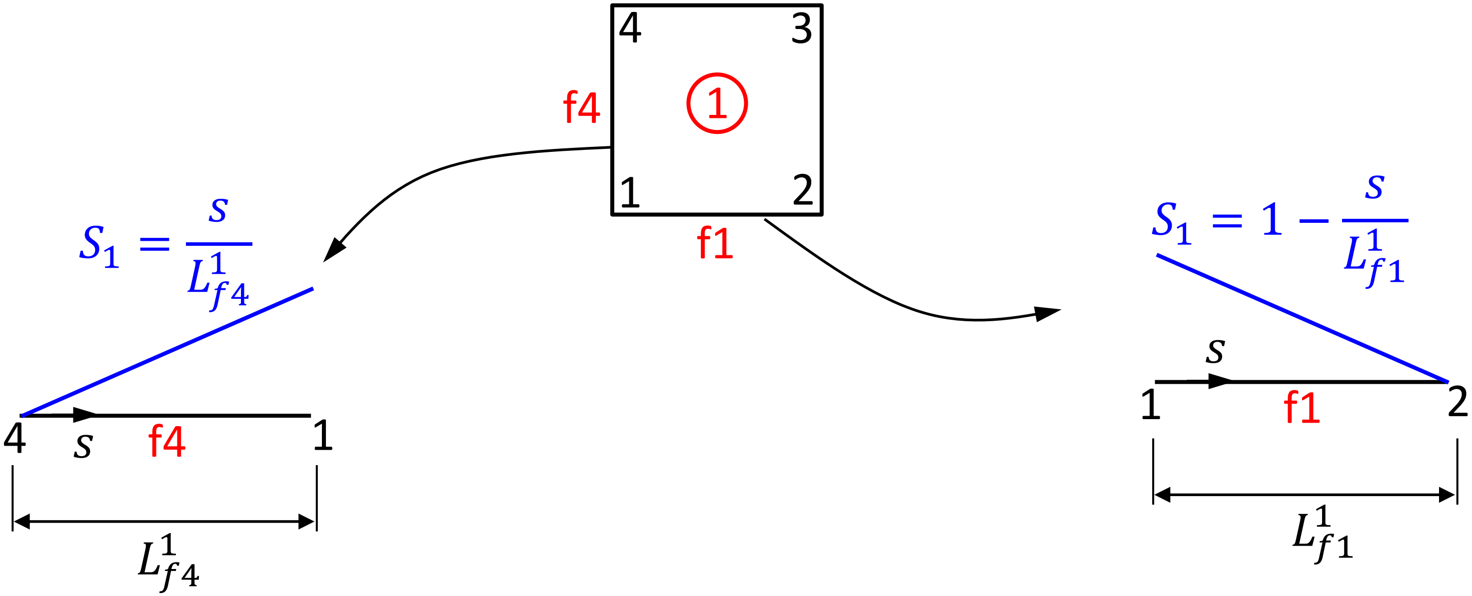
Calculation of $\{Q\}$ (cont'd)

- Consider the common and simple case of q_n = constant.
- Let's study the calculation of

$$Q_1^1 = \int_{f1} S_1 q_n \, ds + \int_{f4} S_1 q_n \, ds$$

- S_1 is a 2D shape function but it reduces to a first order function over faces 1 and 4 of element 1.





Calculation of $\{Q\}$ (cont'd)

$$Q_1^1 = \int_{f1} S_1 q_n \, ds + \int_{f4} S_1 q_n \, ds$$

$$Q_1^1 = \int_{s=0}^{L_{f1}^1} \left(1 - \frac{s}{L_{f1}^1}\right) q_B \, ds + \int_{s=0}^{L_{f4}^1} \frac{s}{L_{f4}^1} q_L \, ds$$

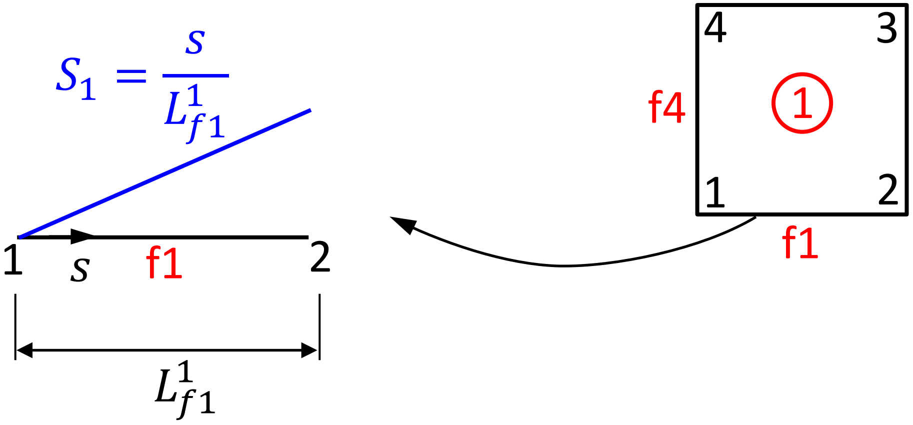
$$\boxed{Q_1^1 = \frac{q_B}{2} L_{f1}^1 + \frac{q_L}{2} L_{f4}^1}$$

- Same procedure can be followed to calculate Q_2^1.

$$Q_2^1 = \int_{f1} S_2 q_n \, ds$$

$$Q_2^1 = \int_{f1} \frac{s}{L_{f1}^1} q_B \, ds$$

$$\boxed{Q_2^1 = \frac{q_B}{2} L_{f1}^1}$$



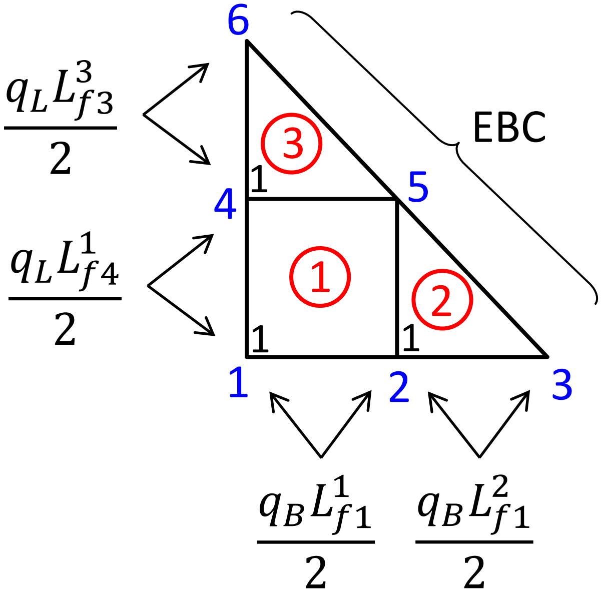
Calculation of $\{Q\}$ (cont'd)

- Calculation of Q_4^1, Q_1^2 and Q_1^3 follow the same procedure.

$$\boxed{Q_4^1 = \frac{q_L}{2} L_{f4}^1 \quad , \qquad Q_1^2 = \frac{q_B}{2} L_{f1}^2 \quad , \qquad Q_1^3 = \frac{q_L}{2} L_{f3}^3}$$

- Summary :

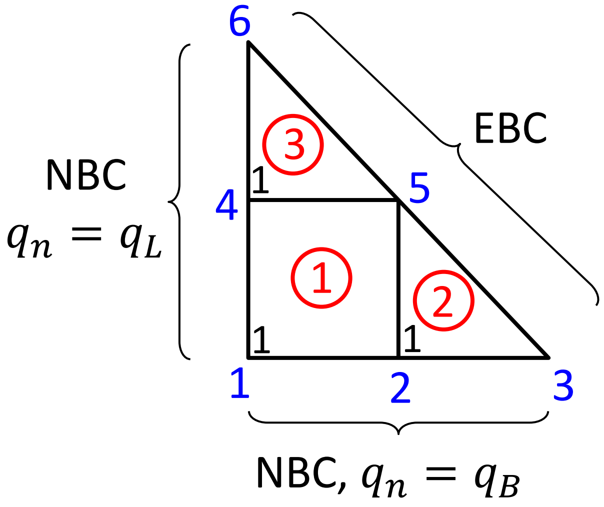
- From the bottom face of e=1, amount of provided SV is $q_B L_{f1}^1$ and it is divided equally to Q_1^1 and Q_2^1.
- From the bottom face of e=2, amount of provided SV is $q_B L_{f1}^2$ and it is divided equally to Q_1^2 and Q_2^2.
- From the left face of e=1, amount of provided SV is $q_L L_{f4}^1$ and it is divided equally to Q_1^1 and Q_4^1.
- From the left face of e=3, amount of provided SV is $q_L L_{f3}^3$ and it is divided equally to Q_1^3 and Q_3^3.



Calculation of $\{Q\}$ (cont'd)

- Assembled $\{Q\}$ vector is

$$Q = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} \frac{q_B}{2} L_{f1}^1 + \frac{q_L}{2} L_{f4}^1 \\ \frac{q_B}{2} L_{f1}^1 + \frac{q_B}{2} L_{f1}^2 \\ Q_3 \\ \frac{q_L}{2} L_{f4}^1 + \frac{q_L}{2} L_{f3}^3 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

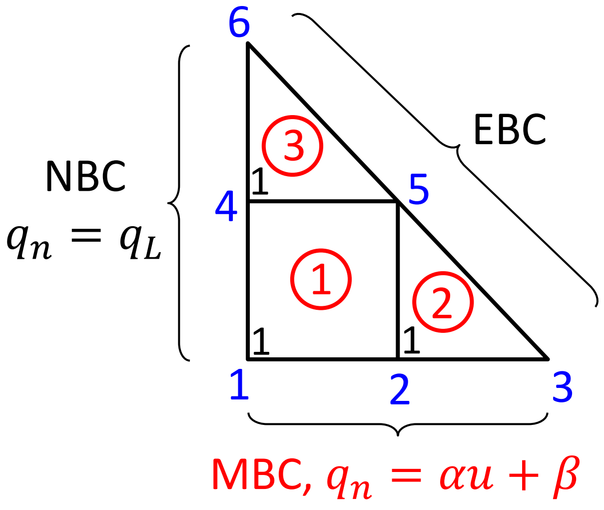


- Note that it is not possible to evaluate Q_3, Q_5 and Q_6 exactly, and these are not necessary due to given EBC for u_3, u_5 and u_6.

Calculation of $\{Q\}$ (cont'd)

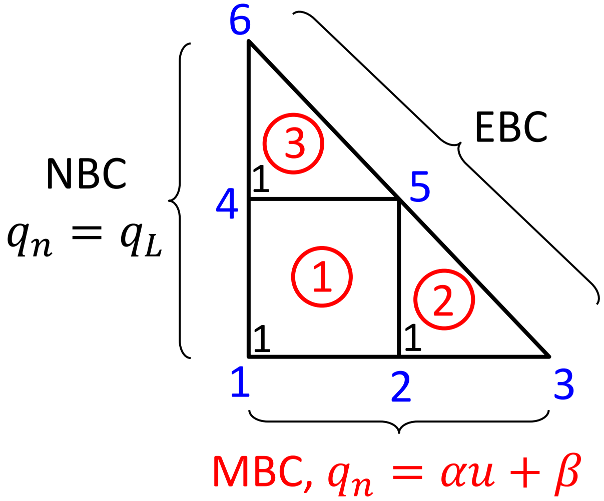
- Question : What if q_n is not constant at an NBC boundary?
- Answer : Just evaluate the line integrals with the given variable q_n.

- Question : What if the BC is not NBC but MBC? Consider the following case where bottom BC is MBC with constant α and β.



Calculation of $\{Q\}$ (cont'd)

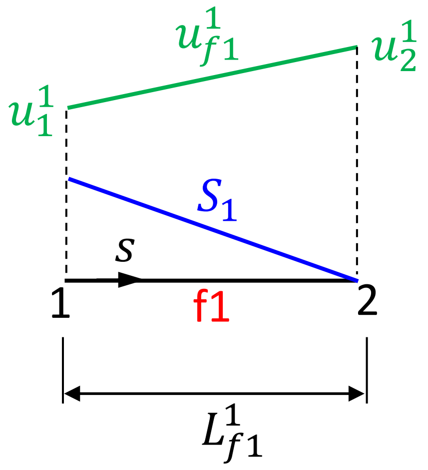
$$Q_1^1 = \underbrace{\int_{f1} S_1 q_n \, ds}_{} + \underbrace{\int_{f4} S_1 q_n \, ds}_{\text{Same as before } = \frac{q_L}{2} L_{f4}^1}$$



$$\int_{f1} \left(1 - \frac{s}{L_{f1}^1}\right)\left(\alpha u_{f1}^1 + \beta\right) ds$$

$$u_{f1}^1 = \left(1 - \frac{s}{L_{f1}^1}\right) u_1^1 + \frac{s}{L_{f1}^1} u_2^1$$

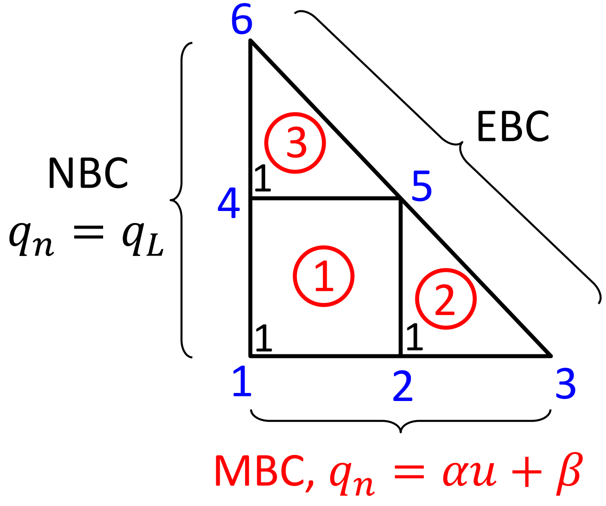
$$u_{f1}^1 = u_1^1 + \frac{u_2^1 - u_1^1}{L_{f1}^1} s$$



Calculation of $\{Q\}$ (cont'd)

$$Q_1^1 = \int_{s=0}^{L_{f1}^1} \left(1 - \frac{s}{L_{f1}^1}\right)\left[\alpha\left(u_1^1 + \frac{u_2^1 - u_1^1}{L_{f1}^1} s\right) + \beta\right] ds + \frac{q_L}{2} L_{f4}^1$$

$$Q_1^1 = \underbrace{\boxed{\frac{\beta}{2} L_{f1}^1 + \frac{\alpha L_{f1}^1}{3} u_1^1 + \frac{\alpha L_{f1}^1}{6} u_2^1}}_{\text{Contribution of the MBC of the bottom face}} + \underbrace{\frac{q_L}{2} L_{f4}^1}_{\text{Contribution of the NBC of the left face}}$$



- To calculate Q_2^1 a similar integral is evaluated but this time with S_2.

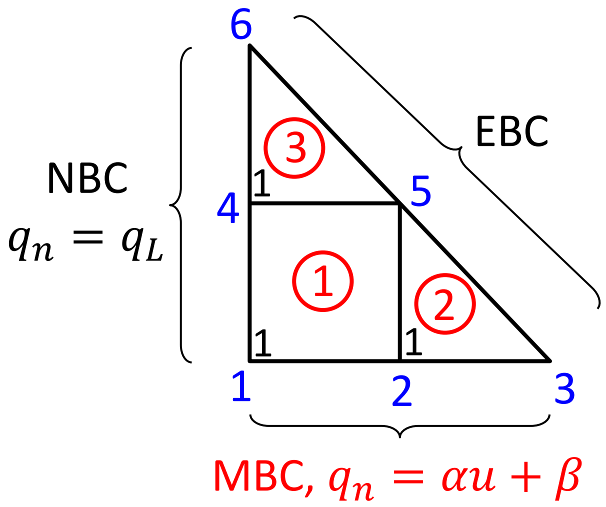
$$Q_2^1 = \int_{s=0}^{L_{f1}^1} \left(\frac{s}{L_{f1}^1}\right)\left[\alpha\left(u_1^1 + \frac{u_2^1 - u_1^1}{L_{f1}^1} s\right) + \beta\right] ds$$

$$\boxed{Q_2^1 = \frac{\beta}{2} L_{f1}^1 + \frac{\alpha L_{f1}^1}{6} u_1^1 + \frac{\alpha L_{f1}^1}{3} u_2^1}$$

Calculation of $\{Q\}$ (cont'd)

- Calculation of Q_1^2 is just the same as Q_1^1.

$$Q_1^2 = \frac{\beta}{2} L_{f1}^2 + \frac{\alpha L_{f1}^2}{3} u_1^2 + \frac{\alpha L_{f1}^2}{6} u_2^2$$



- Assembled $\{Q\}$ is

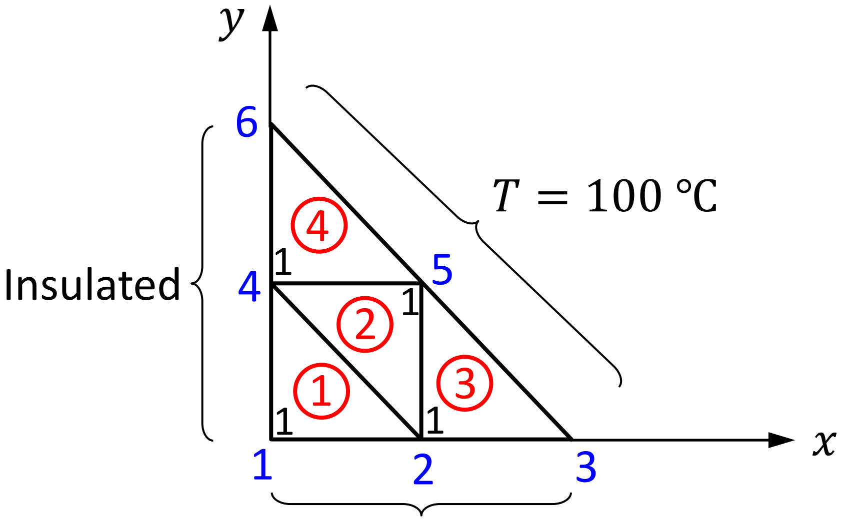
$$Q = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} \frac{\beta}{2} L_{f1}^1 + \frac{\alpha L_{f1}^1}{3} \enclose{circle}{u_1^1} + \frac{\alpha L_{f1}^1}{6} \enclose{circle}{u_2^1} + \frac{q_L}{2} L_{f4}^1 \\ \frac{\beta}{2} L_{f1}^1 + \frac{\alpha L_{f1}^1}{6} \enclose{circle}{u_1^1} + \frac{\alpha L_{f1}^1}{3} \enclose{circle}{u_2^1} + \frac{\beta}{2} L_{f1}^2 + \frac{\alpha L_{f1}^2}{3} \enclose{circle}{u_1^2} + \frac{\alpha L_{f1}^2}{6} \enclose{circle}{u_2^2} \\ Q_3 \\ \frac{q_L}{2} L_{f4}^1 + \frac{q_L}{2} L_{f3}^3 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

Circled terms need to be transferred to the $[K]$ matrix. So it is better to write them using global indices.

Example 5.5

Example 5.5: Determine the temperture distribution over the following 2D geometry. Obtain unknown nodal temperatures. Thermal conductivity of the medium is $1.3\,W/(mK)$.

First local corners of the elements are shown with ‘‘1’’s inside the elements.



$$-k\frac{dT}{dy} = -h(T - T_\infty)$$

$$h = 5\frac{W}{m^2K}, \quad T_\infty = 20\ ℃$$

Node coordinates [m]

Node 1 : (0, 0)
Node 2 : (0.5, 0)
Node 3 : (1, 0)
Node 4 : (0, 0.5)
Node 5 : (0.5, 0.5)
Node 6 : (0, 1)

Example 5.5 (cont'd)

- Governing DE is

$$-\nabla \cdot (k \nabla T) = 0$$

- Elemental weak form (Slide 5-6) is

$$\int_{\Omega^e} k \left(\frac{\partial T}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial w}{\partial y} \right) d\Omega = \int_{\Omega^e} wf \, d\Omega + \oint_{\Gamma^e} w \underbrace{q_n}_{q_n = k\frac{\partial T}{\partial x}n_x + k\frac{\partial T}{\partial y}n_y} d\Gamma$$

$$K_{ij}^e = \int_{\Omega^e} k \left(\frac{\partial S_i^e}{\partial x}\frac{\partial S_j^e}{\partial x} + \frac{\partial S_i^e}{\partial y}\frac{\partial S_j^e}{\partial y} \right) d\Omega$$

$$F_i^e = \int_{\Omega^e} f S_i \, d\Omega$$

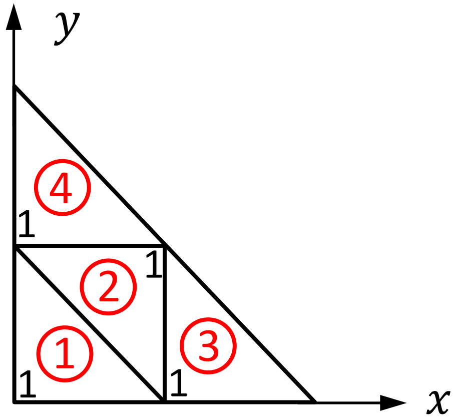
- We can start by calculating the Jacobian matrix of each element.

Example 5.5 (cont'd)

- **e=1 :** $$[J^1] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

  $$|J^1| = 0.25$$

  $$[J^1]^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$



- **e=2 :** $$[J^2] = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \\ 0.5 & 0 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}$$

  $$|J^2| = 0.25$$

  $$[J^2]^{-1} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

- Elements 3 and 4 have the same shape and size as element 1 and their first local node is at the right angle corner. Therefore their Jacobian matrices are the same.

$$[J^3] = [J^4] = [J^1]$$

Example 5.5 (cont'd)

- Elemental systems can now be calculated.

$$K_{ij}^e = \int_{\Omega^e} k \left(\frac{\partial S_i^e}{\partial x} \frac{\partial S_j^e}{\partial x} + \frac{\partial S_i^e}{\partial y} \frac{\partial S_j^e}{\partial y} \right) d\Omega$$

$$\begin{aligned} K_{ij}^e = \int_{\Omega^e} k \Bigg[& \left(\frac{\partial S_i}{\partial \xi} (J^e)_{11}^{-1} + \frac{\partial S_i}{\partial \eta} (J^e)_{12}^{-1} \right) \left(\frac{\partial S_j}{\partial \xi} (J^e)_{11}^{-1} + \frac{\partial S_j}{\partial \eta} (J^e)_{12}^{-1} \right) \\ & + \left(\frac{\partial S_i}{\partial \xi} (J^e)_{21}^{-1} + \frac{\partial S_i}{\partial \eta} (J^e)_{22}^{-1} \right) \left(\frac{\partial S_j}{\partial \xi} (J^e)_{21}^{-1} + \frac{\partial j}{\partial \eta} (J^e)_{22}^{-1} \right) \Bigg] |J^e| d\Omega \end{aligned}$$

$$F_i^e = \{0\}$$

e=1 :

$$K^1 = \begin{bmatrix} 1.3 & -0.65 & -0.65 \\ -0.65 & 0.65 & 0 \\ -0.65 & 0 & 0.65 \end{bmatrix}$$

Example 5.5 (cont'd)

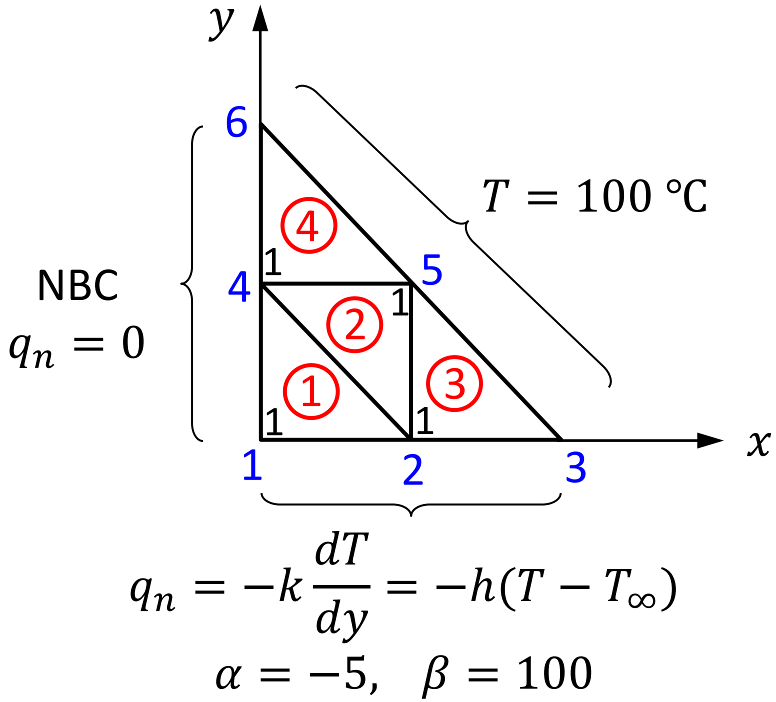
e=2 :

$$K^2 = \begin{bmatrix} 1.3 & -0.65 & -0.65 \\ -0.65 & 0.65 & 0 \\ -0.65 & 0 & 0.65 \end{bmatrix}$$

e=3 and 4 :

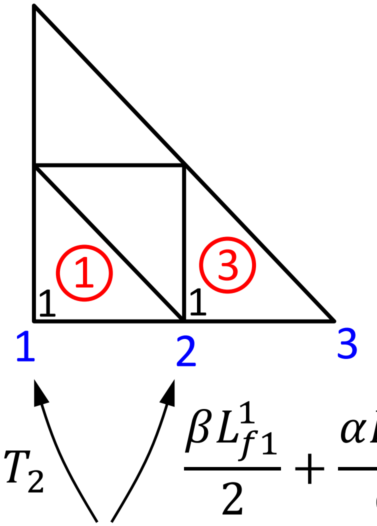
$$[K^3] = [K^4] = [K^1]$$

- Now the $\{Q\}$ vector should be calculated.
- Only contribution will come from the two MBC faces at the bottom.



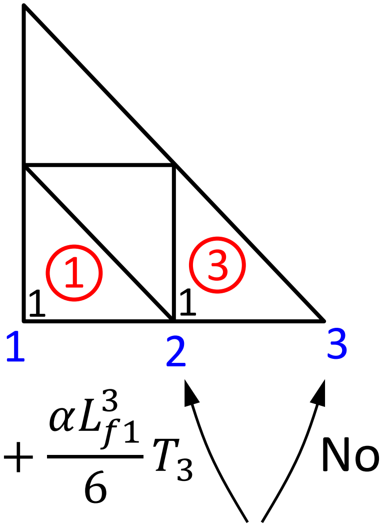
Example 5.5 (cont'd)

Contributions of the MBC at face 1 of e=1 to Q_1 and Q_2.



$$\frac{\beta L_{f1}^1}{2} + \frac{\alpha L_{f1}^1}{3} T_1 + \frac{\alpha L_{f1}^1}{6} T_2 \qquad \frac{\beta L_{f1}^1}{2} + \frac{\alpha L_{f1}^1}{6} T_1 + \frac{\alpha L_{f1}^1}{3} T_2$$

Contribution of the MBC at face 1 of e=3 to Q_2 and Q_3. The contribution to Q_3 is not required because node 3 is an EBC node.



$$\frac{\beta L_{f1}^3}{2} + \frac{\alpha L_{f1}^3}{3} T_2 + \frac{\alpha L_{f1}^3}{6} T_3$$

No need

Example 5.5 (cont'd)

$$Q = \left\{ \begin{array}{c} \frac{100}{2}0.5 + \frac{-5(0.5)}{3}T_1 + \frac{-5(0.5)}{6}T_2 \\ \frac{100}{2}0.5 + \frac{-5(0.5)}{6}T_1 + \frac{-5(0.5)}{3}T_2 + \frac{100}{2}0.5 + \frac{-5(0.5)}{3}T_2 + \frac{-5(0.5)}{6}T_3 \\ Q_3 \\ 0 \quad \longleftarrow \text{Pay attention} \\ Q_5 \\ Q_6 \end{array} \right\}$$

$$Q = \left\{ \begin{array}{c} 25 - 0.8333T_1 - 0.4167T_2 \\ 50 - 0.4167T_1 - 1.6667T_2 - 0.4167T_3 \\ Q_3 \\ 0 \\ Q_5 \\ Q_6 \end{array} \right\}$$

Example 5.5 (cont'd)

- Global system is

$$\begin{bmatrix} 1.3 & -0.65 & 0 & -0.65 & 0 & 0 \\ & 2.6 & -0.65 & 0 & -1.3 & 0 \\ & & 0.65 & 0 & 0 & 0 \\ & & & 2.6 & -1.3 & -0.65 \\ & \text{sym.} & & & 2.6 & 0 \\ & & & & & 0.65 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 25 - 0.8333T_1 - 0.4167T_2 \\ 50 - 0.4167T_1 - 1.6667T_2 - 0.4167T_3 \\ Q_3 \\ 0 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

- Take the unknonws due to MBC from the $\{Q\}$ vector into the $[K]$ matrix.

$$\begin{bmatrix} 2.1333 & -0.2333 & 0 & -0.65 & 0 & 0 \\ -0.2333 & 4.2667 & -0.2333 & 0 & -1.3 & 0 \\ 0 & -0.65 & 0.65 & 0 & 0 & 0 \\ -0.65 & 0 & 0 & 2.6 & -1.3 & -0.65 \\ 0 & -1.3 & 0 & -1.3 & 2.6 & 0 \\ 0 & 0 & 0 & -0.65 & 0 & 0.65 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 50 \\ Q_3 \\ 0 \\ Q_5 \\ Q_6 \end{Bmatrix}$$

Example 5.5 (cont'd)

- Apply reduction for the known T_3, T_5 and T_6.

$$\begin{bmatrix} 2.1333 & -0.2333 & -0.65 \\ -0.2333 & 4.2667 & 0 \\ -0.65 & 0 & 2.6 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 25 \\ 50 + 0.2333(100) + 1.3(100) \\ 0 + 1.3(100) + +0.65(100) \end{Bmatrix}$$

- As seen MBC's do not destroy the symmetry of the reduced system.
- Solve for the unknown primary variables

$$\begin{Bmatrix} T_1 \\ T_2 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 43.3 \\ 50.0 \\ 85.8 \end{Bmatrix} \ ^\circ\mathrm{C}$$

- Constant T lines should be parallel to the EBC boundary and they should be perpendicular to the insulated boundary.